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LETTER TO THE EDITOR

A quantum group version of quantum gauge theories in two dimensions

M Karowski† and R Schrader‡

Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-1000 Berlin, Federal Republic of Germany

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Abstract. For the special case of the quantum group $SL_q(2, \mathbf{C})$ ($q = \exp(\pi i/r)$, $r \geqslant 3$) we present an alternative approach to quantum gauge theories in two dimensions. We exhibit the similarities to Witten's combinatorial approach which is based on the ideas of Migdal. The main ingredient is the Turaev–Viro combinatorial construction of topological invariants of closed, compact 3-manifolds and its extension to arbitrary compact 3-manifolds as given by the authors in collaboration with W Müller.

Based on ideas originally invented by Migdal [6] for the case $\Sigma = \mathbf{R}^2$, Witten [9] gave a combinatorial formulation of quantum gauge theories on an arbitrary compact two-dimensional Riemann manifold (Σ, η) with metric η. (For a path integral approach, see also [2, 3].) It is part of a program devoted to a calculation of the volume of the moduli space of flat connections on compact 2-manifolds.

For the reader's convenience and for later comparison we recall the construction in [9] for the case that Σ is oriented. Let X denote a cell decomposition of Σ (actually a triangulation would suffice) with oriented i-cells c^i ($0 \leqslant i \leqslant 2$). As in standard lattice gauge theories a discrete version of a parallel transport is a map $U(\cdot)$ from the set of oriented 1-cells c^1 into G such that $U(c_1^*) = U(c_1)^{-1}$, where c_1^* equals c_1 but with the opposite orientation. The gauge group is the set of all maps $V(\cdot)$ from the set of zero cells c^0 into G, $c^0 \mapsto V(c^0)$. This gauge group operates on the parallel transporters via $V(\cdot)$: $U(\cdot) \mapsto U_V(\cdot)$ with

$$U_V(c^1) = V(c_i^0)U(c^1)V(c_f^0)^{-1} \tag{1}$$

where c_i^0 and c_f^0 are the initial and final vertices of c^1 respectively. For a given configuration $U(\cdot)$ and an oriented 2-cell c^2, let

$$U(c^2) = \prod_{c^1 \in \partial c^2} U(c^1) \tag{2}$$

be the parallel transporter around c^2, where the product is taken in cyclic order, counter-clockwise say. The conjugacy class of $U(c^2)$ is unique and gauge invariant. For the given metric η on Σ let $\rho(c^2)$ be the area of c^2. Also let χ_α denote a complete set of irreducible characters of G and $c_2(\alpha)$ the value of the Casimir operator in the

† E-mail: karowski@vax1.physik.fu-berlin.dbp.de.
‡ E-mail: schrader@vax1.physik.fu-berlin.dbp.de.

representation α defined by χ_α. With $\dim\alpha=\chi_\alpha(e)$ being the dimension of this representation, set

$$\Gamma(U(c^2),\rho(c^2))=\sum_\alpha \dim\alpha\,\chi_\alpha(U(c^2))\exp(-e^2\rho(c^2)c_2(\alpha)/2). \tag{3}$$

Here $e\neq 0$ is a real parameter, the coupling constant. Now define

$$Z_X(e^2,\eta)=\int\prod_{c^1\in X}\mathrm{d}U(c^1)\prod_{c^2\in X}\Gamma(U(c^2),\rho(c^2)) \tag{4}$$

where $\mathrm{d}U$ is the normalized Haar measure on G. The main result of Witten in this context is that $Z_X(e^2,\eta)$ is independent of the particular choice of X. Thus if X is a triangulation, an elementary argument shows, for example, invariance under Alexander moves [1] and this suffices to establish the claim. Therefore $Z_X(e^2,\eta)$ gives rise to an invariant denoted by $Z_\Sigma(e^2\rho)$ in [9], where ρ is the total area of Σ for given η. By the argument given in [9], $Z_\Sigma(0)$ can be viewed as the volume of the moduli space of flat G-connections on Σ. Furthermore, if $\Sigma=\Sigma^g$ is a Riemann surface of genus g, then

$$Z_{\Sigma^g}(e^2\rho)=\sum_\alpha\frac{\exp(-e^2\rho c_2(\alpha)/2)}{(\dim\alpha)^{2g-2}}. \tag{5}$$

Now $Z_\Sigma(e^2\rho)$ was generalized as follows. Suppose $\Sigma=\Sigma_n$ has n holes, i.e. the boundary $\partial\Sigma$ of Σ consists of components $\partial\Sigma_i=S^1$ $(i=1,\dots,n)$. For the induced cell decomposition ∂X_i of $\partial\Sigma_i$ and given $U(\cdot)$, let $U(\partial X_i)$ be the parallel transport around ∂X_i and set

$$Z_{\Sigma_n}(e^2\rho;\alpha_1,\dots,\alpha_n)=\int\prod_{c^1\in X}\mathrm{d}U(c^1)\prod_{c^2\in X}\Gamma(U(c^2),\rho(c^2))\prod_i\overline{\chi_{\alpha_i}(U(\partial X_i))}. \tag{6}$$

For the particular case where $\Sigma=\Sigma_n^g$ has genus g, this gives

$$Z_{\Sigma_n^g}(e^2\rho;\alpha_1,\dots,\alpha_n)=\delta_{\alpha_1,\dots,\alpha_n}\frac{\exp(-e^2\rho c_2(\alpha)/2)}{(\dim\alpha_1)^{2g-2+n}}. \tag{7}$$

Here $\delta_{\alpha_1,\dots,\alpha_n}=1$ if all α_i are equal and zero otherwise.

Suppose now Σ with area ρ is cut into pieces Σ_μ with areas ρ_μ $(\rho=\Sigma_\mu\rho_\mu)$ by cutting along circles C_i $(i=1,\dots,n)$ labelled by representation α_i and resulting in a colouring $\underline{\alpha}_\mu$ on the boundary circles of Σ_μ. Then one has the surgery formula

$$Z_\Sigma(e^2\rho)=\sum_{\underline{\alpha}}\prod_\mu Z_{\Sigma_\mu}(e^2\rho,\underline{\alpha}_\mu). \tag{8}$$

By cutting Σ^g into $(2g-2)$ spheres, each with three holes, in the standard way, relations (7) and (8) directly lead to (5).

The aim of this letter is to present a quantum group version for the case $\mathrm{SL}_q(2,\mathbf{C})$ $(q=\exp\mathrm{i}\pi/r,\ r\geq 3)$. This has the advantage that one may directly work at $e^2=0$ such that the resulting partition function is metric independent. Note that in (3) the infinite sum is well defined due to the presence of the cut-off $\exp(-e^2\rho(c^2)c_2(\alpha)/2)$. In the $\mathrm{SL}_q(2,\mathbb{C})$ case the parameter r acts as a regularizing factor, since we will only have to deal with representations α labelled as elements in $\mathscr{I}=\{0,\frac{1}{2},\dots,\frac{r}{2}-1\}$. In the classical limit $r\to\infty$ these results agree with those obtained by Witten for the case $e^2=0$ and $\mathrm{G}=\mathrm{SU}(2)$.

The idea is based on the combinatorial construction of Turaev and Viro [7] of topological invariants of closed compact 3-manifolds using $\mathrm{SL}_q(2,\mathbf{C})$ and its extension to arbitrary compact 3-manifolds as given in [4].

In fact, given the Turaev–Viro state sum $Z_{TV}(M)$ for compact 3-manifolds M, we define for any compact 2-manifold Σ the following quantities which are complex numbers

$$Z(\Sigma)=Z_{TV}(\Sigma\times I) \tag{9}$$

and where I is the unit interval. Note that Σ need not be orientable.

We want to show that this simple construction relating two-dimensional theories to three-dimensional ones has interesting features with properties closely related to those given in the preceding section. In fact, in [5], in case the compact 3-manifold M is oriented, we introduced partition functions $Z(M, G_{\underline{\alpha}})$ for coloured graphs $G_{\underline{\alpha}}$ on the boundary ∂M. Roughly speaking a coloured graph is a graph G on ∂M, where to each maximal interval I_i of G one associates an element α_i of $\mathcal{I}$. These partition functions are homotopy invariants of the coloured graphs and are useful in order to obtain surgery relations in analogy to (8). Given this construction, we may extend (9) in case Σ is oriented to define

$$Z(\Sigma, G^l_{\underline{\alpha}}, G^r_{\underline{\beta}})=Z_{TV}(\Sigma\times I, G^l_{\underline{\alpha}}\cup G^r_{\underline{\beta}}) \tag{10}$$

where $G^r_{\underline{\alpha}}$ is a coloured graph on $\Sigma\times\{0\}$ and $G^l_{\underline{\beta}}$ is another coloured graph on $\Sigma\times\{1\}$. We are interested in the special case where again $\Sigma=\Sigma^g_n$ is a Riemann surface of genus g with n holes. Let $G^r_{\underline{\beta}}$ be the empty graph and let $G^l_{\underline{\alpha}}=G_{\underline{\alpha}}$ ($\underline{\alpha}=(\alpha_1,\ldots,\alpha_n)$) be the coloured graph consisting of n circles S^1 around these n holes and carrying the colours $\alpha_1,\ldots,\alpha_n$ respectively.

Then by the methods and results in [5] (see, in particular, lemmas 5.1 and 5.3 and appendix B) one has in analogy to (7)

$$Z(\Sigma^g_n, G_{\underline{\alpha}})=\delta_{\alpha_1,\ldots,\alpha_n}\frac{1}{(w^2_{\alpha_1})^{2g-2+n}} \tag{11}$$

where

$$w^2_\alpha=(2\alpha+1)_{-q}=(-1)^{2\alpha}\frac{q^{2\alpha+1}-q^{-2\alpha-1}}{q-q^{-1}} \tag{12}$$

is the q-dimension associated to the representation α. Obviously w^2_α replaces $\dim\alpha=2\alpha+1$. In fact, $\lim_{r\to\infty}w^2_\alpha=(-1)^{2\alpha}(2\alpha+1)$.

With otherwise the same notation as in (8) it is also easy to prove the same surgery formula

$$Z(\Sigma)=\sum_{\underline{\alpha}}\prod_{\mu}Z(\Sigma_\mu, G_{\underline{\alpha}_\mu}). \tag{13}$$

In particular we obtain

$$Z(\Sigma^g)=\sum_{\alpha}\frac{1}{(w^2_\alpha)^{2g-2}} \tag{14}$$

which compares with (5) and which has a classical limit for $g\geqslant 2$ equal to $\zeta(2g-2)$.

We note that the right-hand side is essentially the Verlinde formula [8]. In fact, the fusion matrices in the present context are given by

$$N^k_{ij}=\begin{cases}1 & \text{if } k\leqslant i+j,\ j\leqslant i+k,\ i\leqslant k+j,\ r-2\geqslant i+j+k\in\mathbf{Z}\\ 0 & \text{otherwise}\end{cases}$$

with $i, j, k \in \mathcal{J}$, such that $N_{ij}^k = N_{ji}^k = N_{kj}^i$. Also all these matrices N^k commute and with

$$N^2 = \sum_k (N^k)^2 \tag{15}$$

we may rewrite (14) in the form (see e.g. [5], appendices A and C for the present context)

$$Z(\Sigma^g) = w^{-2g+2} \operatorname{trace}(N^2)^{2g-2} \tag{16}$$

with $w^2 = \Sigma_\alpha w_\alpha^4$.

This compares with the partition function for the Chern–Simons theory for the group $G = SU(2)$ at level $k = r - 2$

$$Z_{CS}(\Sigma^g \times S^1) = \operatorname{trace}(N^2)^{2g-2}. \tag{17}$$

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Note added in proof. After completion of the manuscript we have been informed by B Ye Rusakov that the above lattice construction attributed to E Witten [9] is already contained in his article [10].

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